

BIFURCATION INTO SPECTRAL GAPS FOR A NONCOMPACT SEMILINEAR SCHRÖDINGER EQUATION WITH NONCONVEX POTENTIAL

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ABSTRACT. This paper shows that the nonlinear periodic eigenvalue problem

$$\begin{cases} -\Delta u + V(x)u - f(x, u) = \lambda u, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

has a nontrivial branch of solutions emanating from the upper bound of every spectral gap of $-\Delta + V$. No convexity condition is assumed. The following result of independent interest is also proven: the direct sum $Y \oplus Z$ in $H^1(\mathbb{R}^N)$ associated to a decomposition of the spectrum of $-\Delta + V$ remains “topologically direct” in the L^p ’s (in the sense that the projections from $Y + Z$ onto Y and Z are L^p -continuous).

INTRODUCTION

The purpose of this paper is to show that the nonlinear periodic eigenvalue problem

$$(1) \quad \begin{cases} -\Delta u + V(x)u - f(x, u) = \lambda u, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

with V, f being \mathbb{Z}^N -periodic in x and f being superquadratic but subcritical, has nontrivial branches of solutions bifurcating from the upper bound of every spectral gap of $-\Delta + V$ on $L^2(\mathbb{R}^N)$.

From now on, let us consider one of these spectral gaps, say $(a, b) \subseteq \rho(-\Delta + V)$. Of course, it is no lack of generality to suppose that $0 \in (a, b)$. This article is inspired from a previous one with M. Willem [11] where it is proven that (1) possesses a nontrivial solution for every $\lambda \in (a, b)$. This approach was subsequently refined by A. Szulkin [6] who was able to pass from $f \in C^1$ to $f \in C^0$. Here we will consider slightly weaker assumptions than in [6], namely

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- (f1) $V \in L^\infty(\mathbb{R}^N)$ and $f \in \mathcal{C}^0(\mathbb{R}^N \times \mathbb{R})$ are 1-periodic in x_k , $1 \leq k \leq N$, and the linear operator

$$D : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) : u \mapsto -\Delta u + V(x)u$$

with domain $\mathbb{D}(D) = H^2(\mathbb{R}^N)$ is invertible (with continuous inverse);

- (f2) there exists $2 < p < 2^* := 2N/(N-2)$ and $c > 0$ such that for all $(x, u) \in \mathbb{R}^N \times \mathbb{R} : |f(x, u)| \leq c(1 + |u|^{p-1})$;
(f3) $f(x, u) = o(|u|)$ uniformly in $x \in \mathbb{R}^N$ as $u \rightarrow 0$;
(f4) there exists $\alpha > 2$ such that : for every $u \in \mathbb{R}$ and every $x \in \mathbb{R}^N$, $0 \leq \alpha F(x, u) \leq f(x, u)u$;
(f5) $\lim_{|u| \rightarrow \infty} \min_{x \in [0, 1]^N} F(x, u) > 0$.

where $F(x, u) := \int_0^u f(x, v) dv$. The proofs given in [6] are still valid under (f1)–(f5) and so there exists a nontrivial solution u_λ of (1) for all $\lambda \in (a, b)$. (See section 2 for more details.) This solution is obtained as a critical point of

$$\mathcal{E}_\lambda : H^1(\mathbb{R}^N) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (V - \lambda)u^2 - \int_{\mathbb{R}^N} F(x, u) dx.$$

Indeed, under (f1)–(f5), \mathcal{E}_λ is well defined on $H^1(\mathbb{R}^N)$ and possesses the linking geometry (see section 2). The main improvement of [11, 6] with respect to previous works on (1) (see [10] and the references therein) is the removal of any convexity condition upon F . However in the latter, it was proved that u_λ actually bifurcates from $(\lambda, u) = (b, 0)$; and so a question raises itself: does this remain true for nonconvex F 's? The question is here settled positively under the additional assumption:

- (f6) there exists a nonnegative \mathbb{Z}^N -periodic function $B \in L^\infty(\mathbb{R}^N) \setminus \{0\}$ and $\beta < 2^*$ such that $F(x, u) \geq B(x)|u|^\beta$ for all $x \in \mathbb{R}^N$ and all u in a neighborhood of 0;

which, together with (f4), may be seen as a local “pinching condition”. A global one was used in [10] (see condition (P), p. 20). Note that (f4) implies $\beta \geq \alpha$. Actually, since a possible $B(x)$ is $\min\{\lim_{u \rightarrow 0} F(x, u)|u|^{-\beta}, 1\}$ and F is periodic, (f6) means that the set of $x \in [0, 1]^N$ satisfying $\lim_{u \rightarrow 0} F(x, u)|u|^{-\beta} > 0$ has nonzero measure. The main theorem of this paper reads as follows.

Theorem 1. *Let (f1)–(f6) hold and (a, b) be the spectral gap of D containing 0. Then, for each $\lambda \in (a, b)$, there exists a nontrivial solution u_λ of (1) such that*

$$\mathcal{E}_\lambda(u_\lambda) = O((b - \lambda)^{\beta/(\beta-2)-N/2}) \rightarrow 0 \quad \text{as } \lambda \rightarrow b.$$

Furthermore, if $\beta < 2 + 4/N$,

$$\|u_\lambda\| = O((b - \lambda)^{1/(\beta-2)-N/4}) \rightarrow 0,$$

where $\|\cdot\|$ denotes the usual norm on $H^1(\mathbb{R}^N)$.

The above condition on β is optimum in the sense that, if $F(x, u) = |u|^\gamma$, then $\alpha \leq \gamma \leq \beta$ so that the best choice for β is $\beta := \gamma$; but $\gamma < 2 + 4/N$ is necessary

for a bifurcation to take place at b when $b = \inf \sigma(D)$ (see [9]). Moreover, as observed by T. Küpper and C.A. Stuart [7, 8], no bifurcation can occur at a . But of course, if f is such that $-f$ satisfies (f1)–(f6), a branch of nontrivial solutions bifurcates from $(\lambda, u) = (a, 0)$ with the convergence rates of theorem 1 (change the signs of λ and u to recover the initial problem).

Before going, in section 2, through the estimates from which bifurcation will eventually result, some preliminary discussion about the spectral properties of the quadratic part \mathcal{Q}_λ of \mathcal{E}_λ is necessary. It is carried out in section 1. In particular, it is said that $H^1(\mathbb{R}^N)$ splits as a direct sum of two closed subspaces Y and Z on which \mathcal{Q}_λ is negative and positive definite respectively. It is of great importance for the projection from $Y + Z$ onto Y (or Z) to be continuous in the L^p 's—and not only in H^1 . This is not the case for every direct sum in H^1 . However appendix A shows that it is true for the particular sum associated to the positive and negative part of the spectrum of $-\Delta + V$. We chose to expound this in an appendix not to interrupt the arguments about bifurcation.

Some natural questions are left unanswered by this paper. First, it would be interesting to know whether the bifurcating branch is continuous. Second, as we said, there is no bifurcation at a . But does any nontrivial solution go to ∞ as $\lambda \rightarrow a$?

Notations. We will write $|u|_p$ for the norm of u in the Lebesgue space $L^p(\mathbb{R}^N)$, $(\cdot|\cdot)_2$ for the inner product in $L^2(\mathbb{R}^N)$, $\|\cdot\|$ for the usual norm on the Sobolev space $H^1(\mathbb{R}^N)$, $\partial\mathcal{F}(u)$ will stand for the Fréchet derivative of the function \mathcal{F} at u , $\mathbb{D}(A)$ for the domain of the operator A , and $B(x, R)$ will denote the open ball in \mathbb{R}^N with center x and radius R .

1. THE QUADRATIC FORM AND BLOCH WAVES

Let $\mathcal{Q}_\lambda : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ be the quadratic form

$$\mathcal{Q}_\lambda(u) := \int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) - \lambda)u^2 \, dx.$$

Since 0 lies in a gap of the spectrum $\sigma(D)$, spectral theory asserts that $H^1(\mathbb{R}^N)$ splits as a direct sum of two closed subspaces Y and Z on which \mathcal{Q}_0 is negative and positive definite respectively:

$$(2) \quad \mathcal{Q}_0(y) \leq -\alpha_0 \|y\|^2, \quad \mathcal{Q}_0(z) \geq \beta_0 \|z\|^2$$

for all $y \in Y$ and $z \in Z$. Moreover, Y and Z are orthogonal in $L^2(\mathbb{R}^N)$, $\mathcal{Q}_0(y + z) = \mathcal{Q}_0(y) + \mathcal{Q}_0(z)$, and the spectral gap is (a, b) with

$$(3) \quad a := \sup_{\substack{y \in Y \\ \|y\|_2=1}} \mathcal{Q}_0(y) < 0 < \inf_{\substack{z \in Z \\ \|z\|_2=1}} \mathcal{Q}_0(z) =: b.$$

The same spectral splitting holds for any $\lambda \in (a, b)$. This is made precise by the following lemma.

Lemma 2. *Let $\lambda \in (a, b)$. Then*

$$\mathcal{Q}_\lambda(y) \leq -\alpha_\lambda \|y\|^2 \quad \text{and} \quad \mathcal{Q}_\lambda(z) \geq \beta_\lambda \|z\|^2$$

for all $y \in Y$ and $z \in Z$, where

$$\alpha_\lambda := \begin{cases} \alpha_0(1 - \lambda/a) & \text{if } \lambda \leq 0, \\ \alpha_0 & \text{otherwise,} \end{cases} \quad \beta_\lambda := \begin{cases} \beta_0 & \text{if } \lambda \leq 0, \\ \beta_0(1 - \lambda/b) & \text{otherwise.} \end{cases}$$

Consequently, $\mathcal{Q}_\lambda(z) - \mathcal{Q}_\lambda(y) \geq N_\lambda \|y + z\|^2$ with $N_\lambda := \frac{1}{2} \min\{\alpha_\lambda, \beta_\lambda\}$.

Proof. We only deal with \mathcal{Q}_λ on Y , the proof on Z being similar. If $\lambda > 0$, $\mathcal{Q}_\lambda(y) \leq \mathcal{Q}_0(y) \leq -\alpha_0 \|y\|^2$. If $\lambda \leq 0$,

$$\mathcal{Q}_\lambda(y) = \mathcal{Q}_0(y) - \lambda |y|_2^2 \leq \mathcal{Q}_0(y) - (\lambda/a) \mathcal{Q}_0(y) \leq -(1 - \lambda/a) \alpha_0 \|y\|^2. \quad \square$$

Since $b \in \sigma(D)$, we know that there exists a Bloch wave Ψ in $H_{\text{loc}}^2(\mathbb{R}^N) \cap \mathcal{C}^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ that satisfies $-\Delta \Psi + V \Psi = b \Psi$ (see [2]). For $R \in (0, +\infty)$, let us set

$$\Psi_R(x) := R^{-N/2} \eta(x/R) \Psi(x)$$

where $\eta \in C_c^\infty(\mathbb{R}^N; [0, 1])$ equals 1 on $B(0, 1)$. Using the fact that Ψ is uniformly almost-periodic in the sense of Besicovich [1], we get the following (see [3, 10]):

$$(B1) \quad \Psi_R \in H^2(\mathbb{R}^N) \cap \mathcal{C}^1(\mathbb{R}^N);$$

$$(B2) \quad \overline{\lim}_{R \rightarrow \infty} \|\Psi_R\| < +\infty;$$

$$(B3) \quad \lim_{R \rightarrow \infty} R^2 \int_{\mathbb{R}^N} |\nabla \Psi_R|^2 + (V - b) \Psi_R^2 \in [0, +\infty);$$

$$(B4) \quad \lim_{R \rightarrow \infty} R^2 \left| -\Delta \Psi_R + (V - b) \Psi_R \right|_2^2 \in [0, +\infty);$$

$$(B5) \quad \lim_{R \rightarrow \infty} R^{(\gamma-2)N/2} \int_{\mathbb{R}^N} B(x) |\Psi_R|^\gamma dx \in (0, +\infty) \text{ for all } \gamma \in [1, +\infty);$$

$$(B6) \quad \|\Psi_R\|_\infty = O(R^{-N/2}).$$

The following consequences of (B3)–(B4) will be used in place of them:

$$(B'3) \quad \mathcal{Q}_b(\Psi_R) = O(R^{-2}) \text{ as } R \rightarrow \infty;$$

$$(B'4) \quad \|\partial \mathcal{Q}_b(\Psi_R)\|^2 = O(R^{-2}) \text{ as } R \rightarrow \infty.$$

Let P be the projector onto Y and $Q = \mathbb{1} - P$ the projector onto Z . For $\lambda \in (a, b)$, let us define

$$\zeta_\lambda := Q \Psi_{R(\lambda)} \in Z$$

with $R(\lambda) := (b - \lambda)^{-1/2}$. The following holds:

Lemma 3. *When $\lambda \rightarrow b$, we have, for all $\gamma \in [2, 2^*]$,*

$$\overline{\lim} \|\zeta_\lambda\| < +\infty; \quad \mathcal{Q}_\lambda(\zeta_\lambda) = O(b - \lambda);$$

$$\underline{\lim} (b - \lambda)^{-(\gamma-2)N/4} \int_{\mathbb{R}^N} B(x) |\zeta_\lambda|^\gamma dx > 0;$$

$$\zeta_\lambda \in L^\infty(\mathbb{R}^N) \text{ and } |\zeta_\lambda|_\infty = O((b - \lambda)^{N/4}).$$

Proof. Since $R \mapsto \Psi_R$ is bounded near ∞ and Q is continuous, $\lambda \mapsto \zeta_\lambda$ is bounded near b .

All along the rest of this proof, we will write R for $R(\lambda)$. When λ is close to b , it follows from the coercivity of $-\mathcal{Q}_\lambda$ on Y that

$$2\alpha_0 \|P\Psi_R\|^2 \leq -2\mathcal{Q}_\lambda(P\Psi_R) = -\langle \partial\mathcal{Q}_\lambda(\Psi_R), P\Psi_R \rangle \leq \|\partial\mathcal{Q}_\lambda(\Psi_R)\| \|P\Psi_R\|$$

and so $\|P\Psi_R\| = O(\|\partial\mathcal{Q}_\lambda(\Psi_R)\|)$. But $\partial\mathcal{Q}_\lambda(\Psi_R) = \partial\mathcal{Q}_b(\Psi_R) + (b - \lambda)O(|\Psi_R|_2)$. Thus, using (B2), we infer $\partial\mathcal{Q}_\lambda(\Psi_R) = O(b - \lambda)$ and

$$\|P\Psi_R\| = O(b - \lambda) \quad \text{as } \lambda \rightarrow b.$$

The second estimate follows from

$$\mathcal{Q}_\lambda(\zeta_\lambda) = \mathcal{Q}_\lambda(\Psi_R - P\Psi_R) = \mathcal{Q}_b(\Psi_R) + (b - \lambda)|\Psi_R|_2^2 - \mathcal{Q}_\lambda(P\Psi_R) = O(b - \lambda).$$

As for the third one, it suffices to note $(\int B|\zeta_\lambda|^\gamma)^{1/\gamma} \geq |(\int B|\Psi_R|^\gamma)^{1/\gamma} - (\int B|P\Psi_R|^\gamma)^{1/\gamma}|$ and to use (B5) and $\int B|P\Psi_R|^\gamma = O(\|P\Psi_R\|^\gamma) = O((b - \lambda)^\gamma) = o((b - \lambda)^{(\gamma-2)N/4})$.

Finally, the last assertion follows from proposition 7 (appendix A) and (B6). Indeed $\Psi_R \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and the restriction of Q to $H^1 \cap L^\infty$ ranges in $H^1 \cap L^\infty$ and is L^∞ -continuous. \square

2. BIFURCATION

First of all, we shall explain the minimax construction that gives a critical point u_λ for all $\lambda \in (a, b)$. Let us define the minimax value:

$$c_\lambda := \inf_{t \geq 0} \sup_{\eta_\lambda(t, M_\lambda)} \mathcal{E}_\lambda$$

where $\eta_\lambda(t, u)$ is the flow generated by some pseudogradient vector field approximating $-\nabla \mathcal{E}_\lambda$ (see [6, 11]) and let M_λ be the set

$$M_\lambda := \{y + s\zeta_\lambda : y \in Y, s \geq 0, \|y + s\zeta_\lambda\| \leq \rho_\lambda\}$$

with ρ_λ large enough such as $\sup_{\partial M_\lambda} \mathcal{E}_\lambda < 0$ where ∂M_λ is the boundary of M_λ in $Y \oplus \mathbb{R}\zeta_\lambda$. Under slightly stronger assumptions than (f1)–(f5), it is proven in [6] that there exists a Palais-Smale sequence at level c_λ and that, for any such Palais-Smale sequence (u_n) , there exists a sequence of translations $(k_n) \subseteq \mathbb{Z}^N$ such that $(u_n(\cdot - k_n))_n$ possesses a subsequence that weakly converges to a nonzero critical point of \mathcal{E}_λ .

This conclusion remains valid under (f1)–(f5). Let us quickly explain why. First, we keep having (1.7) of [6] that reads

$$\forall \delta > 0, \exists c_1 > 0, \quad F(x, u) \geq c_1|u|^\alpha - \delta|u|^2 \text{ on } \mathbb{R}^N \times \mathbb{R}.$$

Indeed, for large u 's, say $|u| \geq \rho$, (f4) and (f5) imply that

$$F(x, u) \geq c_2|u|^\alpha$$

and c_1 can be taken small enough so that $c_1 \leq c_2$ and $c_1|u|^\alpha - \delta|u|^2 \leq 0 \leq F(x, u)$ for all $(x, u) \in \mathbb{R}^N \times [-\rho, \rho]$. As a consequence, \mathcal{E}_λ possesses the so called “linking geometry”. The existence of a $(PS)_{c_\lambda}$ -sequence then follows—for this part relies only on the above geometry and the weak continuity of $u \mapsto \partial \mathcal{E}_\lambda(u)$. Finally, inequalities (1.10) and (1.11) of [6] need not $f(x, u)u$ to be positive but only nonnegative—see eq. (10) below. So, any $(PS)_{c_\lambda}$ -sequence contains a subsequence that weakly converges, up to translations, to a nonzero critical point.

For all $\lambda \in (a, b)$, let $u_\lambda \neq 0$ be such a limit point. We will show that u_λ bifurcates from $(\lambda, u) = (b, 0)$. Let us start with some estimates of the energy of u_λ .

Proposition 4. *Let assumptions (f1)–(f5) hold. Then*

$$(1) \quad 0 \leq \mathcal{E}_\lambda(u_\lambda) \leq c_\lambda.$$

If in addition (f6) is assumed and $\beta < 2 + 4/N$, we have

$$(2) \quad c_\lambda = O((b - \lambda)^{\beta/(\beta-2)-N/2}) \rightarrow 0 \text{ as } \lambda \nearrow b.$$

Proof. (1) Let λ be fixed. Since $\mathcal{E}_\lambda(u) - \frac{1}{2}\partial \mathcal{E}_\lambda(u)u = \int \frac{1}{2}f(x, u) - F(x, u) \geq (\frac{\alpha}{2} - 1) \int F(x, u)$, it is clear that any critical point of \mathcal{E}_λ occurs at a nonnegative level.

Let (u_n) be a Palais-Smale sequence at level c_λ such that $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^N)$. The limit u_λ is a critical point of \mathcal{E}_λ . Let us define

$$\mu_\infty := \lim_{R \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \int_{|x| > R} \frac{1}{2}f(x, u_n)u_n - F(x, u_n) dx.$$

It is clear that $\mu_\infty \geq 0$ and moreover, taking in account that $H^1(\mathbb{R}^N)$ is compactly embedded in all $L_{\text{loc}}^r(\mathbb{R}^N)$ for $2 < r < 2^*$, one readily proves that (see e.g. [12]):

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{1}{2}f(x, u_n)u_n - F(x, u_n) dx = \int_{\mathbb{R}^N} \frac{1}{2}f(x, u_\lambda)u_\lambda - F(x, u_\lambda) dx + \mu_\infty.$$

This can be rewritten as

$$c_\lambda = \mathcal{E}_\lambda(u_n) - \frac{1}{2}\langle \partial \mathcal{E}_\lambda(u_n), u_n \rangle + o(1) = \mathcal{E}_\lambda(u_\lambda) - \frac{1}{2}\langle \partial \mathcal{E}_\lambda(u_\lambda), u_\lambda \rangle + \mu_\infty,$$

which implies $c_\lambda \geq \mathcal{E}_\lambda(u_\lambda)$.

(2) It follows from the very definition of c_λ that it is bounded above by $\sup\{\mathcal{E}_\lambda(y + s\zeta_\lambda) : y \in Y, s \geq 0\}$. Assumption (f6) tells us that there exists some $r > 0$ such that

$$F(x, u) \geq B(x)|u|^\beta \quad \text{for all } |u| \leq r \text{ and } x \in \mathbb{R}^N.$$

Now (f4) says that $u \mapsto F(x, u)|u|^{-\alpha}$ is nondecreasing on $[0, +\infty)$ and nonincreasing on $(-\infty, 0]$, and so

$$F(x, u) \geq F(x, \pm r)r^{-\alpha}|u|^\alpha \geq \kappa_1 B(x)|u|^\alpha \quad \text{for all } |u| \geq r \text{ and } x \in \mathbb{R}^N,$$

where $\kappa_1 := r^{\beta-\alpha}$. Consequently,

$$F(x, u) \geq \kappa_2 B(x) \min\{|u|^\beta, |u|^\alpha\} \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{R}$$

with $\kappa_2 := \min\{1, \kappa_1\}$. Let H a convex function given by lemma 9 (appendix B). Then,

$$(4) \quad \mathcal{E}_\lambda(y + s\zeta_\lambda) \leq \frac{1}{2}\mathcal{Q}_\lambda(y) + \frac{1}{2}s^2\mathcal{Q}_\lambda(\zeta_\lambda) - \kappa_2\varphi(y + s\zeta_\lambda)$$

where $\varphi(u) := \int B(x)H(u)dx$. Using lemma 9 (ii) and (v), we infer that, for all $u, v \in H^1(\mathbb{R}^N)$, $\varphi(u + v) \leq \frac{1}{2}(\varphi(2u) + \varphi(2v)) \leq 2^{\beta-1}(\varphi(u) + \varphi(v))$ and consequently, for all u, w ,

$$\varphi(u + w) \geq 2^{1-\beta}\varphi(u) - \varphi(w)$$

(remember φ is even). Lemma 9 (v) together with this inequality imply

$$(5) \quad \begin{aligned} \varphi(y + s\zeta_\lambda) &\geq \min\{s^\beta, s^\alpha\} \varphi(\zeta_\lambda + y/s) \\ &\geq \min\{s^\beta, s^\alpha\} (2^{1-\beta}\varphi(\zeta_\lambda) - \varphi(y/s)). \end{aligned}$$

On the other hand, since $\sup_{y \in Y, s \geq 0} \mathcal{E}_\lambda(y + s\zeta_\lambda) > 0$, we may as well just take the supremum on the (y, s) 's that satisfy $\mathcal{E}_\lambda(y + s\zeta_\lambda) \geq 0$ and $s > 0$. Using successively $F \geq 0$, lemma 2, and $\varphi(u) \leq |B|_\infty \min\{|u|^\beta, |u|^\alpha\} \leq \kappa_3 \min_{\gamma=\alpha, \beta} \|u\|^\gamma$, we get

$$(6) \quad \begin{aligned} \mathcal{E}_\lambda(y + s\zeta_\lambda) \geq 0 &\Rightarrow \mathcal{Q}_\lambda(y) + s^2\mathcal{Q}_\lambda(\zeta_\lambda) \geq 0 \\ &\Rightarrow \|y/s\|^2 \leq \mathcal{Q}_\lambda(\zeta_\lambda)/\alpha_\lambda \\ &\Rightarrow \varphi(y/s) \leq \kappa_3 \min_{\gamma=\alpha, \beta} (\mathcal{Q}_\lambda(\zeta_\lambda)/\alpha_\lambda)^{\gamma/2}. \end{aligned}$$

Taking account of $\mathcal{Q}_\lambda(y) \leq 0$ and (4)–(6), we get

$$\begin{aligned} \mathcal{E}_\lambda(y + s\zeta_\lambda) &\leq \frac{1}{2}s^2\mathcal{Q}_\lambda(\zeta_\lambda) - \kappa_2 \min\{s^\beta, s^\alpha\} (2^{1-\beta}\varphi(\zeta_\lambda) - \varphi(y/s)) \\ &\leq \max_{\gamma=\alpha, \beta} \frac{1}{2}s^2\mathcal{Q}_\lambda(\zeta_\lambda) - \kappa_2 s^\gamma \Phi_\lambda \end{aligned}$$

where

$$\Phi_\lambda := 2^{1-\beta}\varphi(\zeta_\lambda) - \kappa_3 \min_{\gamma=\alpha, \beta} (\mathcal{Q}_\lambda(\zeta_\lambda)/\alpha_\lambda)^{\gamma/2}$$

and thus, provided that $\Phi_\lambda > 0$ (see below),

$$(7) \quad \mathcal{E}_\lambda(y + s\zeta_\lambda) \leq \kappa_4 \max_{\gamma=\alpha, \beta} \mathcal{Q}_\lambda(\zeta_\lambda)^{\gamma/(\gamma-2)} \Phi_\lambda^{-2/(\gamma-2)},$$

and $\kappa_4 := \max\{(\frac{1}{2} - \frac{1}{\gamma})(\gamma\kappa_2)^{-2/(\gamma-2)} : \gamma = \alpha, \beta\}$.

Let $\lambda \xrightarrow{<} b$. Then $\alpha_\lambda = \alpha_0$ and $\mathcal{Q}_\lambda(\zeta_\lambda) = O(b - \lambda)$, so that

$$(8) \quad \min_{\gamma=\alpha, \beta} (\mathcal{Q}_\lambda(\zeta_\lambda)/\alpha_\lambda)^{\gamma/2} = O((b - \lambda)^{\beta/2}).$$

By lemma 3, $|\zeta_\lambda|_\infty$ is bounded, say by r . Lemma 9 (iii) implies there exists some $\kappa_5 > 0$ such that $H(u) \geq \kappa_5 |u|^\beta$ for $|u| \leq r$. Consequently, one can infer

$$\varphi(\zeta_\lambda) \geq \kappa_5 \int B |\zeta_\lambda|^\beta \geq \kappa_6 (b - \lambda)^{(\beta-2)N/4}$$

which, together with (8), yields

$$\lim_{\lambda \rightarrow b} (b - \lambda)^{-(\beta-2)N/4} \Phi_\lambda > 0,$$

because $(b - \lambda)^{\beta/2} = o((b - \lambda)^{(\beta-2)N/4})$. This, incidentally, implies that $\Phi_\lambda > 0$. It then suffices to plug the estimates of $\mathcal{Q}_\lambda(\zeta_\lambda)$ and Φ_λ in equation (7) to obtain

$$c_\lambda = O\left(\max_{\gamma=\alpha,\beta} (b - \lambda)^{(\gamma-\delta)/(\gamma-2)}\right) \quad \text{where } \delta := \frac{1}{2}N(\beta - 2).$$

To get the desired result, simply note that $\alpha/(\alpha - 2) - \frac{1}{2}N(\beta - 2)/(\alpha - 2) \geq \beta/(\beta - 2) - N/2 > 0$ whenever $\beta < 2 + 4/N$. \square

Theorem 5. Assume (f1)–(f6). Then, when $\lambda \rightarrow b$, $\|u_\lambda\| = O(\sqrt{c_\lambda/N_\lambda})$, and in particular

$$\|u_\lambda\| = O((b - \lambda)^{1/(\beta-2)-N/4}) \rightarrow 0$$

if $\beta < 2 + 4/N$.

Remark 6. 1) N_λ is defined in lemma 2.

2) The above conclusion is slightly stronger than the one of theorem 9.6 in [10]. The latter indeed states (under some additional assumptions) that $\lim_{n \rightarrow \infty} (b - \lambda_n)^{-\theta} \|u_n\| = 0$ for all $0 \leq \theta < 1/(\beta - 2) - N/4$, where $\lambda_n < b$ is a suitable sequence converging to b , and u_n is a critical point of \mathcal{E}_{λ_n} .

Proof. Using proposition 4 and (f4), we infer

$$\begin{aligned} c_\lambda &\geq \mathcal{E}_\lambda(u_\lambda) = \mathcal{E}_\lambda(u_\lambda) - \frac{1}{2} \langle \partial \mathcal{E}_\lambda(u_\lambda), u_\lambda \rangle = \int \frac{1}{2} f(x, u_\lambda) u_\lambda - F(x, u_\lambda) dx \\ (9) \quad &\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right) \int f(x, u_\lambda) u_\lambda dx \end{aligned}$$

Assumptions (f2)–(f3) imply the existence of a constant κ_1 such that $|f(x, u)| \leq \kappa_1 |u|$ if $|u| \leq 1$ and $|f(x, u)| \leq (\kappa_1 |u|)^{p-1}$ if $|u| \geq 1$. Because $f(x, u)u \geq 0$, this yields

$$(10) \quad f(x, u)u = |f(x, u)| |u| \geq \begin{cases} \kappa_1^{-1} |f(x, u)|^2 & \text{if } |u| \leq 1; \\ \kappa_1^{-1} |f(x, u)|^{p'} & \text{if } |u| \geq 1. \end{cases}$$

where $p' := p/(p - 1)$ is the conjugate exponent to p . Fix λ and set $\Gamma := \{x \in \mathbb{R}^N : |u_\lambda(x)| \leq 1\}$. Inequality (9) can be rewritten

$$c_\lambda \geq \kappa_2 \left(\int_\Gamma f(x, u_\lambda) u_\lambda dx + \int_{\mathbb{R}^N \setminus \Gamma} f(x, u_\lambda) u_\lambda dx \right)$$

Combining this with (10), we get

$$(11) \quad \begin{aligned} f_0 &:= \left(\int_{\Gamma} |f(x, u_{\lambda})|^2 dx \right)^{1/2} \leq \kappa_3 c_{\lambda}^{1/2} \\ f_{\infty} &:= \left(\int_{\mathbb{R}^N \setminus \Gamma} |f(x, u_{\lambda})|^{p'} dx \right)^{1/p'} \leq \kappa_3 c_{\lambda}^{1/p'} \end{aligned}$$

with $\kappa_3 := \max\{(\kappa_1 \kappa_2^{-1})^{1/2}, (\kappa_1 \kappa_2^{-1})^{1/p'}\}$. Let us write $u_{\lambda} = y_{\lambda} + z_{\lambda}$ with $y_{\lambda} \in Y$, $z_{\lambda} \in Z$. Lemma 2 and $\partial \mathcal{E}_{\lambda}(u_{\lambda}) = 0$ imply

$$\begin{aligned} \beta_{\lambda} \|z_{\lambda}\|^2 + \alpha_{\lambda} \|y_{\lambda}\|^2 &\leq \mathcal{Q}_{\lambda}(z_{\lambda}) - \mathcal{Q}_{\lambda}(y_{\lambda}) \\ &= \frac{1}{2} \langle \partial \mathcal{Q}_{\lambda}(u_{\lambda}), z_{\lambda} - y_{\lambda} \rangle \\ &= \frac{1}{2} \int f(x, u_{\lambda})(z_{\lambda} - y_{\lambda}) dx \\ &= \frac{1}{2} \int f(x, u_{\lambda}) u_{\lambda} dx - \int f(x, u_{\lambda}) y_{\lambda} dx \end{aligned}$$

and then, using (9) and (11),

$$\begin{aligned} \beta_{\lambda} \|z_{\lambda}\|^2 + \alpha_{\lambda} \|y_{\lambda}\|^2 &\leq \kappa_4 c_{\lambda} + \int_{\mathbb{R}^N} |f(x, u_{\lambda})| |y_{\lambda}| \\ &\leq \kappa_4 c_{\lambda} + f_0 \|y_{\lambda}\|_2 + f_{\infty} \|y_{\lambda}\|_p \\ &\leq \kappa_4 c_{\lambda} + \kappa_5 (c_{\lambda}^{1/2} + c_{\lambda}^{1/p'}) \|y_{\lambda}\| \\ &\leq \kappa_4 c_{\lambda} + \frac{2}{\alpha_{\lambda}} \kappa_5^2 (c_{\lambda}^{1/2} + c_{\lambda}^{1/p'})^2 + \frac{\alpha_{\lambda}}{2} \|y_{\lambda}\|^2 \end{aligned}$$

for some $\kappa_4, \kappa_5 > 0$ independent of λ . Thus, moving $\frac{1}{2} \alpha_{\lambda} \|y_{\lambda}\|^2$ to the left-hand side, we have

$$\frac{1}{2} N_{\lambda} \|u_{\lambda}\|^2 \leq \kappa_4 c_{\lambda} + (2\kappa_5^2 / \alpha_{\lambda}) (c_{\lambda}^{1/2} + c_{\lambda}^{1/p'})^2.$$

Now let $\lambda \rightarrow b$. Therefore $\alpha_{\lambda} = \alpha_0$, $c_{\lambda} \rightarrow 0$, so that $c_{\lambda}^{1/p'} = O(c_{\lambda}^{1/2})$ and

$$N_{\lambda} \|u_{\lambda}\|^2 = O(c_{\lambda}).$$

The second estimate of $\|u_{\lambda}\|$ is obtained by plugging the estimate of c_{λ} of proposition 4 into the first one and using the fact $\lim_{\lambda \rightarrow b} (b - \lambda) / N_{\lambda} < +\infty$. The positivity of $1/(\beta - 2) - N/4$ is equivalent to $\beta < 2 + 4/N$. \square

APPENDIX A. SPECTRAL DECOMPOSITION OF $-\Delta + V$

In this appendix, we will show that the splitting $Y \oplus Z$ of $H^1(\mathbb{R}^N)$ introduced in section 1 remains a direct sum in the $L^p(\mathbb{R}^N)$'s for $2 \leq p \leq 2^*$, in the sense that $\text{cl}_{L^p} Y \cap \text{cl}_{L^p} Z = \{0\}$ with cl_{L^p} denoting the closure in $L^p(\mathbb{R}^N)$. We will start with the following stronger proposition.

Proposition 7. *Let $H^1(\mathbb{R}^N) = Y \oplus Z$ where Y (resp. Z) is the negative (resp. positive) eigenspace of D in H^1 , and $P : H^1 \rightarrow H^1$ (resp. $Q = \mathbb{1} - P$) be the projector onto Y (resp. Z) parallel to Z (resp. Y). Then, for any $p \in [1, +\infty]$, the restrictions of P and Q to $H^1 \cap L^p$ range in $H^1 \cap L^p$ and are L^p -continuous.*

Proof. If $\tilde{P}, \tilde{Q} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ denote the projectors on the negative and positive eigenspaces of D in L^2 respectively, it is well known (see e.g. [10], section 8) that $P = \tilde{P}|_{H^1}$ and $Q = \tilde{Q}|_{H^1}$. So it is sufficient to prove the proposition for $L^2(\mathbb{R}^N)$, \tilde{P} , and \tilde{Q} instead of $H^1(\mathbb{R}^N)$, P , and Q .

Denote $L^p(\mathbb{R}^N; \mathbb{C}) = L^p(\mathbb{R}^N) + iL^p(\mathbb{R}^N)$ the complexification of $L^p(\mathbb{R}^N)$ and let D_p be the operator

$$\begin{aligned} D_p : L^p(\mathbb{R}^N; \mathbb{C}) &\rightarrow L^p(\mathbb{R}^N; \mathbb{C}) : u \mapsto -\Delta u + V(x)u \\ \mathbb{D}(D_p) &:= \{u \in L^p(\mathbb{R}^N; \mathbb{C}) : D_p u \in L^p(\mathbb{R}^N; \mathbb{C})\}. \end{aligned}$$

It is proven in [4] that the spectrum $\sigma(D_p) \subseteq \mathbb{R}$ is independent of $p \in [1, +\infty]$ and moreover, for any $\lambda \notin \sigma(D_p) = \sigma(D_2) = \sigma(D)$,

$$(12) \quad (D_p - \lambda)^{-1} = (D_2 - \lambda)^{-1} \quad \text{on } L^p(\mathbb{R}^N; \mathbb{C}) \cap L^2(\mathbb{R}^N; \mathbb{C}).$$

Then $0 \notin \sigma(D_p)$ and we may speak of the (eigen)projectors P_p, Q_p on the negative and positive eigenspaces of D_p . Since $\sigma(D_p)$ is bounded below, the projector P_p may be defined as follows: if Γ is a right-oriented curve around the negative part of $\sigma(D_p)$ (but not crossing the spectrum), then (see [5]):

$$P_p = \frac{1}{2i\pi} \int_{\Gamma} (D_p - \lambda)^{-1} d\lambda.$$

Accordingly, (12) yield

$$P_p = P_2 \quad \text{on } L^p(\mathbb{R}^N; \mathbb{C}) \cap L^2(\mathbb{R}^N; \mathbb{C}).$$

That concludes the proof because $\tilde{P} = P_2|_{L^2(\mathbb{R}^N)}$ (and $\tilde{Q} = \mathbb{1} - \tilde{P}$). \square

Corollary 8. *Let $Y \oplus Z$ be the splitting of $H^1(\mathbb{R}^N)$ according to the positive and negative part of $\sigma(D)$. Then, for all $p \in [2, 2^*]$,*

$$L^p(\mathbb{R}^N) = \text{cl}_{L^p} Y \oplus \text{cl}_{L^p} Z.$$

Proof. Let P and Q be the projectors of proposition 7. Since P, Q are L^p -continuous and $\mathcal{C}_c^\infty(\mathbb{R}^N) \subseteq Y + Z$ is dense in $L^p(\mathbb{R}^N)$, P and Q extend to continuous projectors P_p and Q_p on $L^p(\mathbb{R}^N)$ which leave invariant $\text{cl}_{L^p} Y$ and $\text{cl}_{L^p} Z$ respectively. From $PZ = \{0\}$ we infer $P_p(\text{cl}_{L^p} Z) = \{0\}$. Thus

$$\text{cl}_{L^p} Y \cap \text{cl}_{L^p} Z = \{0\}.$$

Now let $u \in L^p(\mathbb{R}^N)$. By density there exists a sequence $(u_n) \subseteq \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $u_n \rightarrow u$ in L^p . By continuity,

$$P_p u_n \xrightarrow{L^p} P_p u \in \text{cl}_{L^p} Y, \quad Q_p u_n \xrightarrow{L^p} Q_p u \in \text{cl}_{L^p} Z,$$

and so $u = P_p u + Q_p u \in \text{cl}_{L^p} Y + \text{cl}_{L^p} Z$. \square

APPENDIX B. EXISTENCE OF A CONVEX LOWER BOUND

This appendix is devoted to some elementary calculus showing the existence of a convex lower bound of $\min\{|u|^\beta, |u|^\alpha\}$ with the same asymptotic behavior.

Lemma 9. *Let $\beta \geq \alpha > 2$. There exists an even function $H \in \mathcal{C}^1(\mathbb{R}; [0, +\infty))$ such that*

- (i) *for all $u \in \mathbb{R}$, $H(u) \leq \min\{|u|^\beta, |u|^\alpha\}$;*
- (ii) *H is convex;*
- (iii) *$\lim_{u \rightarrow 0} H(u) |u|^{-\beta} = 1$;*
- (iv) *$\lim_{|u| \rightarrow \infty} H(u) |u|^{-\alpha} = 1$;*
- (v) *for all $u \in \mathbb{R}$ and $t \geq 0$, $\min\{t^\alpha, t^\beta\} H(u) \leq H(tu) \leq \max\{t^\alpha, t^\beta\} H(u)$.*

Remark 10. As consequences of the above facts, we get $H(0) = 0$, $\partial H(0) = 0$, and $H(u) > 0$ for all $u \neq 0$.

Proof. Let $h \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ be the odd function defined by $h(u) := \min\{\beta|u|^{\beta-1}, \alpha|u|^{\alpha-1}\}$ for $u \geq 0$ and $H(u) := \int_0^u h$. The map $G(u) := \min\{|u|^\beta, |u|^\alpha\}$ is \mathcal{C}^1 on $\mathbb{R} \setminus \{\pm 1\}$. Let ∂G be its derivative. It is clear that $G(u) = \int_0^u \partial G$ for all $u \in \mathbb{R}$. Then ((i)) follows from $h(u) \leq \partial G(u)$ for all $u \in [0, +\infty) \setminus \{1\}$ and the evenness of H . Since h is increasing, H is (strictly) convex. An immediate computation shows

$$H(u) = \begin{cases} |u|^\beta & \text{if } |u| \leq \rho, \\ \kappa + |u|^\alpha & \text{otherwise,} \end{cases}$$

for some $0 < \rho \leq 1$ and $\kappa := \rho^\beta - \rho^\alpha$. That proves the asymptotic behaviors of H . Finally, the definition of h implies that, for $t \geq 0$ and $u \geq 0$,

$$\min\{t^{\alpha-1}, t^{\beta-1}\} h(u) \leq h(tu) \leq \max\{t^{\alpha-1}, t^{\beta-1}\} h(u)$$

and then ((v)) follows by integrating and taking into account the evenness of H . \square

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